Reliability-based Optimization considering Design Variables of Discrete Size

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Abstract

This contribution presents an approach for performing reliability-based optimization (RBO) with emphasis on the treatment of discrete design variables. The key idea of the proposed approach is the construction of an approximate representation of the structural reliability (i.e. a meta-model for reliability). This allows replacing the original RBO problem with a series of approximate yet accurate subproblems. These approximate subproblems can be solved most efficiently using any appropriate optimization method. Application examples involving the design of steel frames illustrate the features of the proposed approach.

Key words: reliability-based optimization, discrete design variables, meta-model, sequential approximations, advanced simulation methods, linear expansion

1. Introduction

Design processes in engineering aim at developing systems that satisfy prescribed performance requirements, e.g. serviceability and collapse conditions. For example, in the case of a pedestrian bridge, the level of vibration must be limited in order to offer sufficient comfort to the users. In practical situations, a number of parameters which affect the performance are of an uncertain nature such as, e.g. loadings, deterioration processes, etc. Under such circumstances, it is not possible to ensure that the performance requirements will be fulfilled during the whole life time of a system, as the uncertainty in the aforementioned parameters propagates to the system’s response. Probability theory offers an appropriate framework for modeling uncertainty, as it allows calculating the reliability associated with a specific design solution, i.e. the probability that the performance remains within an acceptable range. Besides producing reliable systems, design processes also aim at providing economical solutions, as resources are always scarce. However, reliability and economy are competing objectives, i.e. highly reliable systems are usually associated with high construction and maintenance costs. Hence, the problem of designing a system can be treated within the framework of reliability-based optimization (RBO) \cite{11, 17, 20, 28, 40, 45, 49, 52, 59, 60, 63}. In this way, it is possible to determine an appropriate trade off between costs and reliability. Nonetheless, and in spite

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of its evident advantages over deterministic design procedures, the application of RBO in engineering practice has remained limited in the past. This is due to the fact that for solving a RBO problem, it is necessary to evaluate the system’s response for different values of the design variables and several realizations of the uncertain parameters, thus involving high numerical costs.

In practical problems, it is often the case that design variables must be chosen within a discrete manufacturer’s inventory [43], e.g. in the case of steel structures, profiles must be selected from a set of discrete elements. This issue poses an additional challenge for solving RBO problems, as discrete optimization can be considerably more involved than its continuous counterpart [27, 56]. Therefore, the objective of this contribution is proposing a framework for performing RBO considering discrete design variables in an efficient way. In particular, the solution scheme proposed in this paper integrates a sequential approximations strategy (see, e.g. [1, 39]) with an approximate, explicit representation of reliability of a system w.r.t. discrete design variables. It is important to indicate that the approach proposed in this contribution refers to discrete design variables describing sizes of structural members. Therefore, design variables modeling, e.g. type of construction materials are not included within the scope this contribution.

Deterministic optimization procedures involving discrete design variables have been thoroughly studied in the literature. In fact, this type of problem has been categorized in 6 different classes [27], based on the presence of continuous and discrete variables, the possibility that a discrete variable assumes non-discrete values during the optimization process, etc. The different algorithms for solving this type of problems have been classified in three groups [61]; the first one includes branch and bound methods, which can be numerically demanding (see, e.g. [27]); the second group refers to approximation methods using, e.g. dual formulations (see, e.g. [50, 51]), discrete Lagrangian method (see, e.g. [30]), sequential approximations (see, e.g. [1, 39, 43]), etc.; the third group includes the so-called ad-hoc methods, such as annealing approaches (see, e.g. [16]), evolutionary optimization algorithms (see, e.g. [9, 23, 62]), etc. In addition, meta-models of the structural response can also be used to reduce computational costs, see e.g. [26, 46, 56].

Procedures for RBO considering discrete design variables have not been addressed as frequently as their deterministic counterpart. In most studies, ad-hoc optimization algorithms have been integrated directly with a reliability method. For example, in [10, 25], the problem of optimization under uncertainty has been approached by means of Genetic Algorithms (GA - see, e.g. [21]) and direct Monte Carlo Simulation (MCS - see, e.g. [42]) for reliability analysis; as both the optimization and the reliability algorithms require a large number of function calls, numerical costs can be very high. A variant of this approach is the application of the so-called Population based simulation algorithm [24], in which GA and MCS are applied simultaneously. As a means for reducing computational efforts related with reliability analysis, the application of optimization methods in combination with the First Order Reliability Method (FORM - see, e.g. [14]) was investigated in, e.g. [22, 57, 65]. Another ap-
proach for reducing computational efforts is the application of meta-models, such as Artificial Neural Networks (see, e.g. [44, 45]).

This contribution introduces two innovative aspects in comparison with similar publications in the field. In the first place, the application of advanced simulation methods (see, e.g. [54]) for reliability assessment allows solving RBO problems involving a large number of random variables (in the order of thousands). In the second place, this contribution proposes a novel meta-model for reliability analysis. The basis of this meta-model is constructing a linear expansion of the function related with the performance of a structural system; this is an extension of a procedure originally developed by the authors for RBO problems involving continuous design variables [29, 66]. The most salient feature of the aforementioned meta-model for reliability is that it can be constructed using a single reliability analysis plus some additional structural analyses; this is a most desirable feature, as usually reliability analyses are numerically demanding.

The structure of this paper is as follows. In Section 2, the mathematical formulation of the reliability-based optimization problem considering discrete design variables and the general solution strategy are presented. Section 3 introduces the aforementioned novel approach for constructing a meta-model of the reliability. Implementation issues are discussed in Section 4. Finally, two application examples involving the optimal design of steel frames are presented in Section 5.

2. Formulation and Solution Strategy

2.1. Mathematical Formulation of the RBO Problem

The mathematical formulation of the problem under study is the following [36].

\[
\begin{align*}
\min_{\mathbf{y} \in \Omega_y} & \left\{ C_0(\mathbf{y}) + \sum_{i=1}^{n_P} C_i(\mathbf{y}) p_{F,i}(\mathbf{y}) \right\} \\
\text{subject to} & \\
 p_{F,i}(\mathbf{y}) & \leq p_{F,i}^{tol}, \quad i = 1, \ldots, n_P \\
 h_j(\mathbf{y}) & \leq 0, \quad j = 1, \ldots, n_D \\
y_k & \in \Omega_{y,k} = \{ y_{k,1}, y_{k,2}, \ldots, y_{k,e_k} \}, \quad k = 1, \ldots, n_y 
\end{align*}
\]

In eq. (1), \( \mathbf{y} \) groups the design variables in a vector of \( n_y \) components; the design variables can assume values in the discrete set \( \Omega_y \); \( C_i(\cdot) \), \( i = 0, \ldots, n_P \) are deterministic cost functions depending on the design variables; \( p_{F,i}(\cdot) \), \( i = 1, \ldots, n_P \) represent the probability that the \( i \)-th performance criterion is not satisfied; \( p_{F,i}^{tol} \), \( i = 1, \ldots, n_P \) is the prescribed acceptable failure probability associated with the \( i \)-th performance criterion; \( h_j(\cdot) \), \( j = 1, \ldots, n_D \) are deterministic constraints. Finally, each of the design variables \( y_k, \ k = 1, \ldots, n_y \) must be contained in the discrete set \( \Omega_{y,k} \) (note that this set contains a total of \( e_k \) elements and that \( \Omega_y = \Omega_{y,1} \times \Omega_{y,2} \times \ldots \times \Omega_{y,n_y} \)). Moreover, it is assumed
that for every possible value of a discrete design variable, there is a number of linked properties, e.g. when selecting a specific steel profile, the cross sectional area and the second moments of inertia are uniquely defined.

RBO problems can be formulated in several different ways, see e.g. [17]. Thus, the formulation presented in eq. (1) is a special case that was selected as it serves the purposes of this contribution. It should be noted that the objective function in eq. (1) does not refer only to damage (collapse) states, but it also might include costs due to construction and partial damage states [20, 36].

The probabilities \( p_{F,i}, i = 1, \ldots, n_P \) are calculated by means of the classical multidimensional integral:

\[
p_{F,i}(y) = \int_{D_i(y, \theta) \geq 1} f(\theta) d\theta
\]

where \( \theta \) is the vector grouping uncertain parameters present in the model (characterized by a probability density function \( f(\cdot) \)) and where \( D_i(\cdot, \cdot) \) is the so-called normalized demand function [5, 66]. This function is defined in such way that it is equal or larger than 1 whenever a particular combination of the vector of design variables and uncertain parameters lead to the violation of the \( i \)-th performance criterion and it can be interpreted as a demand to capacity ratio. A typical example of \( D_i(\cdot, \cdot) \) is the ratio between stress at a certain location and the ultimate stress.

It should be noted that the integral in eq. (2) can be alternatively written as:

\[
p_{F,i}(y) = P[D_i(y, \theta) \geq 1]
\]

where \( P[\cdot] \) denotes probability of occurrence of the expression between brackets.

In problems of practical interest, the integral in eq. (2) must be evaluated numerically, as closed-form solutions exist in a few, very simple cases only. Moreover, the dimensionality of the vector of uncertain parameters can be very large, e.g. in the order of thousands [31, 54]. Thus, simulation methods are best suited for assessing probabilities [53]. In particular, advanced simulation methods such as Horseracing Simulation [32], Line Sampling [34], Subset Simulation [5], etc. are particularly advantageous for solving problems in structural reliability. In spite of the fact that these methods are much more efficient than direct MCS, the numerical efforts required to apply advanced simulation methods are not negligible.

2.2. Solution Strategy

The direct solution of the problem in eq. (1) by means of an appropriate numerical optimization method implies that the space of discrete design variables must be explored for several different combinations of \( y \). For each of these combinations, it is necessary to evaluate probabilities, which in turn implies the evaluation of the normalized demand function (which is related with the structural response) for a fixed value of the design variables and several different realizations of the uncertain
parameters. In view of the fact that the evaluation of the structural response may imply the computation of large, numerically involved FE models, the solution of a RBO problem using a direct strategy is limited to academic examples only, as numerical costs associated with realistic engineering problems may grow unboundedly.

A possible means for reducing the numerical efforts associated with the solution of RBO problems is the introduction of approximation concepts. In particular, in this contribution, the concept of meta-model for the structural reliability is applied. Such an idea has been used a number of times in the literature of RBO considering continuous design variables, see e.g. [12, 20]. The advantage of this approach is that the reliability assessment step is decoupled from the optimization problem, i.e. the meta-model is comparatively inexpensive to evaluate and thus, any appropriate algorithm can be used to solve the optimization problem. As the construction of this approximation over the entire domain of the design variables can be demanding, it may be easier to generate an approximation of the failure probabilities over a local domain \( \Omega^L_y \subset \Omega_y \) [28].

The local meta-model of the failure probabilities is applied for solving the target RBO problem within the framework of sequential approximations [1, 39]. That is, the original optimization problem in eq. (1) is broken into a series of subproblems. Each of this optimization subproblems is solved in a local domain and the local optimum is used as a starting point for a new iteration of the algorithm. In mathematical terms, an optimization subproblem has the following form.

\[
\min_{y \in \Omega^L_y} \left\{ C_0(y) + \sum_{i=1}^{n_P} C_i(y) \overline{p}_{F,i}(y) \right\}
\]

subject to

\[
\overline{p}_{F,i}(y) \leq p_{F,i}^{tol}, \quad i = 1, \ldots, n_P
\]

\[
h_j(y) \leq 0, \quad j = 1, \ldots, n_D \tag{4}
\]

In eq. (4), \( \overline{p}_{F,i}(\cdot) \) represents a local approximation of the probability around the \( r \)-th candidate optimal design and \( \Omega^L_y \) is the associated \( r \)-th local domain. By solving this subproblem several times, it is possible to generate a series of candidate optimal designs \( y^r, \quad r = 1, \ldots, n_L \). These candidate optimal designs may or may not converge to the optimal solution of the original RBO problem. That is, the series of candidate designs may converge towards a global optimum, a local optimum or may not even converge towards a local optimum. In fact, different starting points \( (y^1) \) may lead to different design solutions [39]. The reasons explaining this possible behavior of the optimization strategy are the approximations introduced and the inherent challenges associated with the solution of optimization problem involving discrete variables [1]. However, these properties associated with the application of sequential approximations do not impose – in principle – a limitation. On one hand, the candidate optimal designs which are generated through the sequential approximations should represent an improvement over the initial one, thus providing most valuable information. On the other hand,
engineering criteria and the knowledge on the problem at hand provide guidelines for assessing the quality of the approximate optimum that is obtained. Different criteria can be used to stop the optimization algorithm based on sequential approximations, see e.g. [1]. A simple yet effective criterion is, e.g. in case the best candidate optimal design that has been determined does not improve within the last $n$ iterations (e.g. $n = 4$), the optimization is stopped. Such criterion is used here for the sake of simplicity; however, other criteria could be used as well.

For a better understanding of the optimization scheme using sequential approximations, consider the schematic representation in fig. (1), where a RBO problem with $n_y = 2$ and $n_{e,1} = n_{e,2} = 9$ is depicted in the design variable space. Starting from the initial design $y^1$, optimization is performed in the local domain $\Omega_{y,1}$ and the new candidate optimal design $y^2$ is determined. Then, a new local domain $\Omega_{y,2}$ is defined in order to find $y^3$.

In this contribution, Genetic Algorithms (GA - see, e.g. [21]) are applied to solve the optimization subproblems. Thus, the open issues for applying sequential approximations are how to construct a meta-model of the probabilities and how to define the local optimization domains. These issues are addressed in detail in the next Sections of this contribution.

3. Meta-model for Reliability

The construction of the meta-model for reliability implies two steps. In the first one, standard reliability analysis is performed at the current candidate optimal design $y^r$. In the second step, a linear expansion of the normalized demand is constructed. Each of these steps is discussed in detail below.
3.1. Reliability Analysis at Candidate Optimal Design

As discussed in Section 2.1, the probability of not satisfying the $i$-th performance criterion is calculated by means of the multidimensional integral shown in eq. (2). This probability is the integral of the probability density function associated with the vector of uncertain parameters ($f(\theta)$) over the region where the associated normalized demand function ($D_i(\cdot, \cdot)$) exceeds 1. For constructing a meta-model of the reliability, the first step is determining the probability that the normalized demand exceeds a threshold $b$, i.e.:

$$p_{F,i}(y^r, b) = P[D_i(y^r, \theta) \geq b] = \int_{D_i(y, \theta) \geq b} f(\theta) d\theta$$

where $b \in [1 - \epsilon, 1 + \epsilon]$ and $\epsilon$ is a suitable perturbation, e.g. $\epsilon = 0.3$. Eq. (5) differs from eq. (2) in the following: while the integral in eq. (2) is calculated for a specific value, in eq. (5) the multidimensional integral is calculated for a range of $b$, i.e. the integral in eq. (5) is a function of $b$. The reason for determining this relation will become apparent in the sequence. As already mentioned above, the evaluation of eq. (5) is performed by means of advanced simulation techniques. In particular, the simulation techniques used in this paper are Importance Sampling (IS) using design points [55] and Subset Simulation (SS) [5]. Each of these techniques is briefly described below.

3.1.1. Importance Sampling using Design Points

Importance Sampling (IS) is a simulation technique that allows estimating the multidimensional integral shown in eq. (2). The key concept for its application is the introduction of an importance sampling density (ISD) function $f_{IS}(\theta)$ that is different from zero wherever the original distribution of the uncertain parameters $f(\theta)$ is also different from zero within the failure domain. The ISD should be selected such that if $N$ samples of $\theta$ are generated following the distribution $f_{IS}(\theta)$, a large number of these samples belongs to the event of interest. That is, there are several occurrences of $D_i(y^r, \theta^{(m)}) \geq b = 1$, where $\theta^{(m)}$ are samples of the uncertain parameters involved in the reliability problem distributed according to $f_{IS}(\theta)$ for $m = 1, \ldots, N$. In this way, an unbiased estimator of the failure probability when applying IS is provided by:

$$p_{F,i}(y^r, b = 1) = \int_{D_i(y^r, \theta) \geq b} \frac{f(\theta)}{f_{IS,i}(\theta)} f_{IS,i}(\theta) d\theta \approx \frac{1}{N} \sum_{m=1}^{N} I_i(y^r, \theta^{(m)}) \frac{f(\theta^{(m)})}{f_{IS,i}(\theta^{(m)})}$$

where $I_i(\cdot, \cdot)$ is an indicator function which is equal to 1 if $D_i(y^r, \theta^{(m)}) \geq b = 1$ and zero otherwise. In this contribution, the ISD is constructed around the so-called design point ($\theta_i^*$), as suggested in [55]. The design point is defined as the realization of $\theta$ (projected in the standard normal space) which belongs to the event of interest and with the minimum Euclidean norm with respect to the origin; in
order to determine \( \theta^* \), the following constrained optimization problem must be solved.

\[
\min_{\theta \in \mathbb{R}^n} \sqrt{\theta_1^2 + \theta_2^2 + \ldots + \theta_n^2}
\]

subject to

\[
D_i(y^r, \theta) \geq b = 1
\]

(7)

In the formulation above, it is assumed that the vector of random variables \( \theta \) is composed by standard Gaussian, independent random variables. For those cases where this condition is not fulfilled, an appropriate transformation can be applied over the random variables, such as, e.g. the Nataf’s transformation (see, e.g. [38]). Methods for solving the optimization problem in eq. (7) have been thoroughly documented in the literature, see e.g. [4, 33, 37].

The application of IS will yield the probability \( P[D_i(y^r, \theta) \geq b = 1] \), as indicated in eq. (6). However, for constructing the meta-model for reliability, it is necessary to determine the probability that the normalized demand function \( D_i(y^r, \theta) \) exceeds values of \( b \) within a certain range \( b \in [1 - \epsilon, 1 + \epsilon] \) (see eq. (5)). In order to determine such function, the following approach is followed. Separate reliability analyses for the cases \( P[D_i(y^r, \theta) \geq 1 - \epsilon] \), \( P[D_i(y^r, \theta) \geq 1] \) and \( P[D_i(y^r, \theta) \geq 1 + \epsilon] \) are carried out. Then, the sought relation is determined by interpolation between these data points.

Finally, it is important to indicate that the application of IS in problems involving a large number of random variables can be quite challenging, depending on the particulars of the reliability problem at hand. A discussion on this issue goes outside the scope of this paper. However, this issue has been thoroughly addressed in, e.g. [6, 31]. Furthermore, IS is applied in this contribution for problems involving a small number of random variables, i.e. 20 or less.

3.1.2. Subset Simulation

SS was introduced in [5, 7]. In this advanced simulation technique, the failure domain \( F \) (\( F = \{ \theta \in \Omega_\theta : D_i(y^r, \theta) \geq 1 \} \)) is defined as a sequence of subsets (or intermediate failure events) \( F_p, p = 1, \ldots, P \) such that \( F_1 \supset F_2 \supset \ldots \supset F_P = F \). Thus, the failure probability can be expressed as:

\[
p_F(y^r) = P[F_P] = P[F_1] \prod_{p=1}^{P-1} P[F_{p+1} \mid F_p]
\]

(8)

where \( P[F_{p+1} \mid F_p] \) is the probability of occurrence of the event \( F_{p+1} \) conditioned on the event \( F_p \). The key issue in SS is selecting the intermediate failure events such that the probabilities \( P[F_{p+1} \mid F_p], p = 1, \ldots, P - 1 \) and \( P[F_1] \) are sufficiently large (e.g. \( P[\cdot] \approx 0.1 \)) in order to estimate them using direct MCS. Thus, a small failure probability can be calculated as the product of larger, conditional probabilities. The practical implementation of SS requires an efficient algorithm for generating samples of the uncertain parameters (\( \theta \)) conditioned on an intermediate failure event,
such as the modified Metropolis algorithm [5, 41].

An interesting feature of SS is that it explores the space of uncertain parameters by means of successive subsets $F_p$, $p = 1, \ldots, P$, each of which is of rarer occurrence than the previous one, i.e. $P[F_{p-1}] > P[F_p]$. This implies that SS provides the probability of occurrence of the event $P[D_i(y^r, \theta) > b]$, with $b \in [0, 1 + \epsilon]$. Therefore, a single run of SS suffices for determining the relation in eq. (5).

3.2. Linear Expansion of Normalized Demand

Assume that a local domain $\Omega_{y}^{L,r}$ centered on $y^r$ has been defined. Moreover, assume that a design $y^*$ has been defined such that $y^* \in \Omega_{y}^{L,r}$. Then, the following linear expansion of the normalized demand is introduced.

$$D_i(y^*, \theta) \approx \overline{D}_i(y^*, \theta) = D_i(y^r, \theta) + \sum_{k=1}^{n_y} a_k (q(y^*_k) - q(y^r_k))$$  (9)

In eq. (9), $q(y^*_k)$ represents the numerical value of a property associated with the $y_k$-th design variable, e.g. if $y_k$ identifies a particular steel profile, $q(y_k)$ could denote the cross sectional area associated with that profile; $a_k$, $k = 1, \ldots, n_y$ are constant, real coefficients, which are assumed to be known at this stage (the issue on how to calculate these coefficients is discussed in detail in Section 4).

The expression in eq. (9) can be interpreted as a linear expansion of $D_i(y^*, \theta)$; this expansion was introduced in the context of RBO considering continuous design variables in [29, 66]. The expansion is linear because the difference in the value of the normalized demand between $(y^*, \theta)$ and $(y^k, \theta)$ is linear w.r.t. $q(y^*_k) - q(y^r_k)$, $k = 1, \ldots, n_y$. Although the expansion in eq. (9) will be in most cases only a rough approximation, it has been observed that for the purposes of generating a meta-model for reliability, it suffices. In addition, it is important to note that for the validity of eq. (9), it is required that the behavior of the normalized demand is monotonic with respect to the property $q(y_k)$ and also it is required that the set $\Omega_{y,k}$ is ordered in such way that it is ensured that $q(y_{k,1}) \leq q(y_{k,2}) \leq \ldots \leq q(y_{k,ek})$. This issue is further discussed in Section 4.

3.3. Practical Application of Meta-model

With the concepts developed above, it is possible to construct a suitable meta-model for the reliability. In order to illustrate the application of this meta-model, assume that the relations in eqs. (5) and (9) are known and that the objective is to determine the probability associated with a design $y^* \in \Omega_{y}^{L,r}$. According to the definition of eq. (3), this probability is calculated as follows.

$$p_{F,i}(y^*) = P[D_i(y^*, \theta) \geq 1]$$  (10)
Replacing the approximation introduced in eq. (9) in the last equation yields the following expression:

\[ p_{F,i}(y^*) \approx P \left[ D_i(y^*, \theta) \geq 1 \right] \]

\[ \approx P \left[ D_i(y^r, \theta) \geq 1 - \sum_{k=1}^{n_y} a_k (q(y^*_{ik}) - q(y^r_{ik})) \right] \]

\[ \approx P \left[ D_i(y^r, \theta) \geq b^* \right], \text{ where } b^* = 1 - \sum_{k=1}^{n_y} a_k (q(y^*_{ik}) - q(y^r_{ik})) \] (11)

The evaluation of the last equation is straightforward, as the relation in eq. (5) has been determined by means of reliability analysis. As it can be noted, the evaluation of the probability associated with \( y^* \) is (approximately) equal to evaluating the probability that the normalized demand associated with \( y^r \) exceeds a threshold \( b^* \). In other words, provided that a reliability analysis is carried out at \( y^r \) (see Section 3.1) and that the coefficients \( a_k, k = 1, \ldots, n_y \) are known (see Section 3.2), it possible to estimate the probability associated with \( y^* \) without performing any additional reliability analysis. Thus, eq. (11) provides a suitable meta-model for reliability, as it allows evaluating reliability at any design \( y^* \in \Omega_y^{L,r} \) without any significant computational effort. In turn, this meta-model allows solving the optimization subproblem formulated in eq. (4) using any appropriate optimization algorithm.

4. Implementation Issues

In this section, different issues concerning the practical implementation of the scheme for RBO described above are addressed.

4.1. Definition of a Local Optimization Domain

The scheme for solving RBO problems considering discrete design variables is based on sequential approximations. A key issue in the practical implementation of this scheme is the selection of the local optimization domains \( \Omega_y^{L,r} \), \( r = 1, \ldots, n_L \), where \( n_L \) indicates the total number of local domains. In this contribution, a very simple scheme, based on the concepts outlined in [1] is applied. That is, the local domain is defined around the current candidate optimal design \( y^r \) by introducing a perturbation. For presenting a formal definition of the local domain, suppose that \( y^r \) is such that \( y^r_{ik} = y_{k,i}, k = 1, \ldots, n_y \), i.e. the \( k \)-th component of the \( r \)-th candidate optimal design is equal to the \( i \)-th element of the discrete design set \( \Omega_y^{L,r} \) (cf. eq. (1)). Then, \( \Omega_y^{L,r} \) is defined as the Cartesian product between a total of \( n_y \) basic sets \( \Omega_y^{L,r,k} \), \( k = 1, \ldots, n_y \), as shown below.

\[ \Omega_y^{L,r} = \Omega_y^{L,r,1} \times \Omega_y^{L,r,2} \times \ldots \times \Omega_y^{L,r,n_y} \] (12)

In the equation above, \( \times \) denotes Cartesian product. In addition, each of the \( n_y \) basic sets mentioned above are defined by introducing a perturbation \( \pm w \) around the \( i_k \)-th element of the discrete design.
set $\Omega_{y,k}$, i.e.:

$$
\Omega_{y,k}^L = \{y_{k,v} \in \Omega_{y,k} : v \in \mathbb{N} \wedge v \in \{\max(i_k - w, 1), \ldots, \min(i_k + w, e_k)\}\}, \ k = 1, \ldots, n_y
$$

(13)

where $w$ is the parameter controlling the size of the local domain and $\mathbb{N}$ denotes the set of natural numbers. It should be noted that the functions $\min(\cdot)$ and $\max(\cdot)$ are introduced in order to ensure that all elements of $\Omega_{y,k}^L$ also belong to $\Omega_{y,k}$. Moreover, note that for the sake of simplicity, $w$ is chosen as a single parameter defining the perturbation introduced in the $n_y$ components of the vector of design variables $y$. The selection of the parameter $w$ has a major influence in the implementation and performance of the proposed scheme for solving RBO problems. An inappropriate selection of this parameter may prevent convergence or increase numerical costs unnecessarily. For example, in case the parameter $w$ is too small, a large number of sequential approximations may be required before determining the optimal solution. On the other hand, in case $w$ is too large, it may be more challenging to identify a candidate optimal solution due to the quality of the meta-model for reliability in large domains. Although no analytical formula has been determined, the following qualitative criteria are found to be useful for selecting the parameter $w$. In case the quality of the meta-model for reliability is high (i.e. there is a fair match between the approximate model and full reliability analyses), then $w$ can be relatively large; in the opposite case, a small $w$ would be a better choice, in order to protect the quality of the meta-model. On the other hand, larger values of $w$ are suggested for the initial steps of the sequential optimization; as the iterations associated with the optimization progress (see Section 2.2), the value of $w$ should decrease continuously. According to the numerical experience of the authors, a reasonable choice for $w$ can be 5 for the initial steps of the sequential approximate optimization; for the last steps of the iteration, $w$ can be chosen as 1 or 2.

4.2. Considerations on the Set of Available Discrete Designs

A key issue for the practical application of the meta-model for reliability described in the previous Section is the construction of the approximate representation of the normalized demand function. Due to the shape of this approximation (namely, a linear polynomial), some conditions concerning the discrete set of design variables should be fulfilled in order to ensure that the proposed approach can be applied to a particular problem. These conditions are the following.

1. The behavior of the normalized demand function should be related with the discrete design variables by means of the property $q(y_k)$, $k = 1, \ldots, n_y$.

2. As the linear expansion of the normalized demand function depends on $(q(y_k^*_{k}) - q(y_k^c))$, $k = 1, \ldots, n_y$, the normalized demand should be monotonic w.r.t. the property $q(y_k)$ and the database of available discrete designs $\Omega_{y,k}$ should be sorted according to that property. Moreover, the property $q(y_k)$ should be relatively smooth. That is, no abrupt changes of the property
between two consecutive designs should exist, i.e. \( q(y_{k,i_k}) \) and \( q(y_{k,i_k+1}) \) (where \( i_k \in [1, \epsilon_k] \)) should not differ considerably.

The selection of the property for the linear expansion in eq. (9) depends on the problem at hand. For example, consider the design of a steel truss under some stress or displacement constraints: as the bars of the structure are expected to work under axial forces only, a natural choice would be creating the linear expansion w.r.t. the area of the cross section. On the contrary, in a steel frame, the beam elements work mainly under bending moments and the most appropriate property for the linear expansion would be then the second moment of inertia of the cross section. Thus, engineering criteria may be used in order to determine which property of the design variables is most appropriate for constructing the linear expansion.

Although engineering criteria should be applicable in several situations, there may exist particular cases where the selection of a property of the design variables for the linear expansion is not evident. In these situations, a feasible means for determining the sought property would be performing MCS drawing samples of both the discrete design variables and the uncertain parameters and then, determine the correlations between the normalized demand function and the different properties associated with each design variable. Then, the highest correlation may indicate which property is more suitable for generating the linear expansion. In cases where two (or more) properties show a strong correlation, a possibility would be to create a linear expansion depending on two (or more) properties; such an approach has been applied customarily in structural optimization, see e.g. [64]. Nonetheless, it must be pointed out that the alternatives of performing MCS for determining important properties and the construction of a linear expansion considering more than one important property have not been further investigated, as these issues are outside the scope of this contribution.

4.3. Calibration of Coefficients of Linear Expansion

The linear expansion of the normalized demand provides an approximation of the function \( D_i(y^*, \theta) \). In other words, the linear expansion is a meta-model for the exact function. Taking this into account, the coefficients \( a_k, k = 1, \ldots, n_y \) of the expansion can be determined by means of any appropriate procedure used in meta-modeling. In particular, in this contribution, these coefficients are adjusted in a least square sense. This implies the minimization of square error \( R \) between the exact and approximate models, which are evaluated at a series of realizations of the vector of design variables and uncertain parameters, i.e.:

\[
R = \sum_{s=1}^{N_S} \left( D_i \left( y^{(s)}, \theta^{(s)} \right) - D_i \left( y^r, \theta^{(s)} \right) - \sum_{k=1}^{n_y} a_k \left( q \left( y^{(s)}_k \right) - q(y^r_k) \right) \right)^2 \tag{14}
\]

In eq. (14), the pair of vectors \( \left( y^{(s)}, \theta^{(s)} \right) \), \( s = 1, \ldots, N_S \) are realizations of the vector of design variables and of the uncertain parameters, respectively.
The details of the algorithm for calibration of the coefficients $a_k$, $k = 1, \ldots, n_y$ are described below (for a detailed description of this algorithm, it is referred to [29, 66]).

1. Create a set of $N_S$ samples of the uncertain parameters such that $D_i(y^r, \theta^{(s)}) \approx 1$, $s = 1, \ldots, N_S$.
   It should be noted that at the reliability analysis stage (see Section 3.1), a number of samples of the uncertain parameters and their associated value of normalized demand function are generated. Therefore, for creating the aforementioned set of $N_S$ samples, it suffices selecting those samples that fulfill the condition that their associated normalized demand is close to one. Numerical experience indicates that a suitable number of samples is $N_S = 100$ [29, 66], i.e. 50 samples with a value of normalized demand function larger than one and another 50 samples with a value smaller than 1. Set $k = 1$.

2. Generate a set of $N_S$ realizations of the vector of design variables, i.e. $y^{(s)}$, $s = 1, \ldots, N_S$; each vector $y^{(s)}$ must be equal to $y^r$, except the $k$-th component $y_k^{(s)}$. This component is selected at random from the set $\Omega_{y,k}^L, r^*$, where $\Omega_{y,k}^L, r^*$ is defined as:

\[
\Omega_{y,k}^L, r^* = \{ y_{k,v} \in \Omega_{y,k} : v \in \mathbb{N} \land v \in \{ \max(i_k - w, 1), \ldots, \min(i_k + w, e_k) \} \land v \neq i_k \}, \ k = 1, \ldots, n_y
\]  

The terms in eq. (15) are the same as those defined for eq. (12).

3. Evaluate $D_i(y^{(s)}, \theta^{(s)})$, $s = 1, \ldots, N_S$.

4. Estimate the coefficient $a_k$ in a least squares sense, i.e.:

\[
a_k = \frac{\sum_{s=1}^{N_S} \left( q(y_k^{(s)}) - q(y_{r,k}^r) \right) \left( D_i(y^{(s)}, \theta^{(s)}) - D_i(y^r, \theta^{(s)}) \right)}{\sum_{s=1}^{N_S} \left( q(y_{k,v}^{(s)}) - q(y_{r,v}^r) \right)^2}
\]  

5. In case $k = n_y$, stop the algorithm. Otherwise, set $k = k + 1$ and return to step 2.

The algorithm described above for the calibration of the linear expansion of the normalized demand has shown to be robust in context with the applications studied in this contribution. However, other approaches for calibrating this expansion could be applied as well, e.g. weighted least squares, etc.

Although the procedure described above has been specially developed for the calibration of the coefficients of the linear expansion of the demand function, it also provides some additional information on the selection of the parameter $w$ that controls the local optimization domains. That is, by examining the dispersion of the samples used to calibrate the aforementioned linear expansion, it is possible to determine qualitatively whether or not the selected parameter $w$ is appropriate. In order to illustrate this issue, consider the schematic representation in fig. (2). In this figure, the abscissa represents the values of the property $q(\cdot)$, the ordinate contains the value $(D_i(y^{(s)}, \theta^{(s)}) - D_i(y^r, \theta^{(s)})$) and the dots represent the samples for calibrating the linear expansion. As it can be noted from the figure, the data points can be reasonably represented by a linear relation for $w_1 = 2$. However, for the case where
\( q \left( y_{k,i}^{k-3} \right) \)
\( q \left( y_{k,i}^{k-2} \right) \)
\( q \left( y_{k,i}^{k-1} \right) \)
\( q \left( y_{k,i}^{k} \right) \)
\( q \left( y_{k,i}^{k+1} \right) \)
\( q \left( y_{k,i}^{k+2} \right) \)
\( q \left( y_{k,i}^{k+3} \right) \)

\( \text{value of property } q(\cdot) \)

\[ D_{i}(y^{(s)}, \theta^{(s)}) - D_{i}(y^{(s)}, \theta^{(s)}) \]

Figure 2: Calibration of coefficients of linear expansion of normalized demand

\( w_{2} = 3 \), a linear relation may not be appropriate. Therefore, a reasonable selection of the parameter \( w \) for the example illustrated in fig. (2) would be 2.

5. Examples

5.1. Example 1: Design of a Linear Steel Frame subject to Dynamic Excitation

5.1.1. General Remarks

A 10-storey steel frame is considered in the first example, as shown in fig. (3). The frame is excited by a horizontal ground acceleration of 15 \( s \) of duration, which is modeled as a stochastic process. The objective is to minimize the volume \( (V) \) of the columns of the frame subject to a reliability constraint, i.e. the probability that the structure does not fulfill a prescribed performance objective due to the ground acceleration should be equal or smaller than \( 10^{-3} \). The design variables are the steel profiles of the columns of the frame; 4 design groups are considered. It is assumed that the frame remains within the elastic range.

5.1.2. Description of the Model

Each floor of the model is supported by six steel columns. It is assumed that the columns and beams are axially very stiff and that the mass is concentrated at the floor level. Therefore, the lateral displacement of the building is characterized as a 10 degree-of-freedom system. The mass of each floor of the model is equal to \( m = 150 \times 10^{3} \) kg. The structure is assumed to possess a classical damping, equal to 2\% for all vibration modes. The Young’s modulus of the steel is taken as \( E = 2 \times 10^{11} \text{ Pa} \).
Additionally, the stochastic ground acceleration is modeled as filtered white noise, involving a total of 1501 random variables; the details of this model can be found in Appendix A.

A single failure event is considered, which refers to a first passage problem during the time of analysis; the structural responses to be controlled are the 10 interstorey displacements and the roof displacement. The threshold values are chosen equal to \(6 \times 10^{-3} m\) for the interstorey drift and \(30 \times 10^{-3} m\) for the roof displacement, respectively. The normalized demand function is thus formulated as shown below.

\[
D(y, \theta) = \max_{q=1,\ldots,11} \left( \max_{z=1,\ldots,1501} \left( \frac{r_q(t_z, y, \theta)}{r_q^*} \right) \right)
\]  

(17)

where \(r_q(\cdot, \cdot, \cdot), q = 1, \ldots, 11\) denotes the \(q\)-th structural response (i.e. for \(q = 1, \ldots, 10\), the interstorey displacement associated with the \(q\)-th floor and for \(q = 11\), the roof displacement), \(r_q^*, q = 1, \ldots, 11\) denotes the threshold level for the \(q\)-th structural response and \(t_z, z = 1, \ldots, 1501\) represents the discrete time steps of analysis (for details, see Appendix A).

A total of 4 design variables \(y_k, k = 1, \ldots, 4\) are defined: \(y_1\) involves all the columns of the storeys 1 to 3; \(y_2\), storeys 4 to 6; \(y_3\), 7 and 8 and \(y_4\), storeys 9 and 10. The 4 design variables are chosen among the W-shaped sections [2]; the database used in this contribution includes a total of 160 profiles.

Constraints referring to buckling (global or local) were ignored in this example, as the objective is analyzing the performance of the proposed approach for RBO in exploring a large discrete set. Therefore, the database includes some profiles which are usually not considered for the design of column elements.

The formal description of the RBO problem to be solved is the following.
\[ \begin{align*}
\min & \quad V(y) \\
\text{subject to} & \quad p_F(y) = P[D(y, \theta) \geq 1] \leq 10^{-3} \\
& \quad y_k \in \Omega_{y,k} = \{y_{k,1}, y_{k,2}, \ldots, y_{k,160}\}, \ k = 1, \ldots, 4 
\end{align*} \tag{18} \]

5.1.3. Results

The underlying reliability problem is solved using SS \cite{5, 7}; for each subset, 1000 samples of the uncertain parameters are generated. The application of SS allows thus obtaining the complete relation \(P[D_i(y, \theta) \geq b]\) (cf. eq. (5)), as already discussed in Section 3.1. Concerning the application of the proposed scheme for RBO, the parameter \(w\) (which controls the size of the local optimization domain) was set equal to 5. The linear expansion of the normalized demand function (cf. eqs. (9) and (17)) is performed w.r.t. to the second moment of inertia (principal axis) of the design variables; furthermore, the database of 160 available profiles for the design variables is sorted according to this property. For the calibration of the coefficients of the linear expansion of the normalized demand function, \(N_S\) is selected as 100.

The initial design is set such that \(y^1 = (W12 \times 136, W12 \times 136, W12 \times 120, W12 \times 120)^T\). The results of the optimization process in terms of the evolution of the objective function and failure probability for the different candidate optimal designs are shown in figs. (4) and (5), respectively. Moreover, in Table 1, the details on the initial and final designs obtained are shown.

Table 1 here

According to the results presented in fig. (4), it is noted that the proposed scheme for RBO produces
a series of optimal candidate designs that improves the value of the objective function at each new iteration (with the exception of the fifth and eighth candidates). A similar behavior is observed in fig. (5): starting from an infeasible design, the algorithm is capable of converging towards a design that fulfills the probability constraint. The results shown in Table 1 indicate that starting from an infeasible design, the optimization procedure is capable of finding a design which is not only feasible but also better in terms of the value of the objective function.

In total, only ten iterations are required in order to determine the optimum. For computing each of these iterations a single analysis plus some additional structural analyses suffices, due to the scheme introduced in this contribution for constructing a meta-model for reliability; this is a most remarkable feature. Concerning the numerical costs, the solution of each optimization subproblem requires 3000 or 4000 structural analyses for assessing reliability plus 400 analyses for calibrating the coefficients of the linear expansion of the normalized demand function.

5.2. Example 2: Design of a Steel Frame considering Seismic Risk

5.2.1. General Remarks

For the second example, the optimal design of a 5-storey steel frame considering expected construction costs and partial damage costs due to seismic risk is considered. A schematic representation of the frame model is shown in fig. (6). The design variables are the cross section of the columns, grouped in 4 sets: the first, second and third sets involve all columns of storeys 1, 2 and 3, respectively, while the fourth design group involves the columns of the storeys 4 and 5. Each of the design variables can be chosen from a discrete set of 25 columns of profiles W14 [2]. Therefore, the discrete set of design variables involves a total of $25^4 = 390625$ possible combinations.

Concerning the quantification of partial damage due to seismic risk, a major difficulty is that the structural performance during the life time is highly uncertain. In order to overcome this difficulty,
efforts have devoted during the last decade for the development of a new framework for design under seismic risk. This new framework has been denoted as Performance-based Design (PBD - see, e.g. [3, 13, 35, 48, 58]). PBD aims at introducing new approaches to conceive structures which exhibit a more predictable and reliable behaviour under common and extreme loading events (e.g., wind loading, earthquakes). The key point of PBD is producing a rational measure of the performance of a structure during its life time. In this way, the owner can take an informed decision about the features of a particular facility based not only in the construction cost but in the maintenance and possible repair costs due to future seismic events. Due to the inherent uncertainties of loadings acting on structures and the imprecise knowledge on the properties and behaviour of construction materials, it is impossible to characterize the performance of a structure using a deterministic approach. Thus, it has been widely acknowledged that the application of probabilistic tools is a necessary condition within the PBD framework [15]; this necessity has been considered in recent works in the area. For example, in the methodology proposed in [3], the characterization of seismic hazard considers probability concepts; in turn, the output of the PBD methodology - losses due to seismic risk, either in economic terms, deaths or downtime - is also characterized in probabilistic terms.

As the output of the PBD procedure is expressed in probabilistic terms, tools for optimization under uncertainty have been recognized as feasible means for assisting engineers in the decision-making process of designing a structure (see, e.g. [8, 15]). In particular, RBO has been identified as a viable approach for design of structural systems subject to seismic risk, see e.g. [47, 69]. Therefore, the objective of this example is investigating the application of RBO for the optimal design of the steel frame in fig. (6) in view of seismic risk. In particular, the aim is minimizing the costs of construction and damage due to seismic events. In order to quantify the structural response, non linear static analysis is applied [18]. Moreover, damage and economical losses are accounted for by means of a simplified procedure introduced in [67, 68].
5.2.2. Formulation of the Problem

As mentioned above, the objective is to minimize the expected costs associated with construction and partial damage due to seismic risk of the steel frame in fig. (6). Under the assumption that the occurrence of significant seismic events can be modeled as a Poisson process (with occurrence rate $\nu$) and that the structure is retrofitted to its original condition after a major earthquake, it is possible to estimate the cost function by means of the following expression, as demonstrated in [67].

$$E[C(y, \theta)] = C_C(y) + \frac{\nu}{\lambda} \left(1 - e^{-\lambda T}\right) + \sum_{i=1}^{n_F} C_i p_{F,i}(y) + \frac{C_m}{\lambda} \left(1 - e^{-\lambda T}\right)$$

In eq. (19), $E[\cdot]$ represents expected value, $C(\cdot, \cdot)$ is the total cost function associated with a facility during the life time $T$, $C_C(y)$ represents the costs of construction, $\lambda$ is the discount rate, $p_{F,i}(\cdot)$ is the probability of occurrence of the $i$-th failure criterion (note that the cost associated with the occurrence of this criterion is $C_i$) and $C_m$ represents the maintenance costs. It should be noted that a failure criterion corresponds to a damage state, e.g. loss of serviceability, minor damage, near collapse, etc.

According to the PBD methodology discussed in [3], the quantification of damage should be performed using fragility curves. However, in this contribution, a simplified approach proposed in [68] for the design of steel structures is applied. In this approach, the storey drift ratio is related with the level of damage, thus allowing to calculate the probability of occurrence of a particular damage state. Details about this approach are presented in Appendix B.

The mathematical formulation of the RBO problem to be solved is the following.

$$\min \ E[C(y, \theta)]$$

subject to

$$y_k \in \Omega_{y,k} = \{y_{k,1}, y_{k,2}, \ldots, y_{k,25}\}, \ k = 1, \ldots, 4$$

5.2.3. Description of the Model

The uncertainty in the seismic hazard is characterized by modeling the peak ground acceleration (pga) as a random variable. In particular, the annual exceedance probability of the pga w.r.t. a prescribed threshold is modeled by means of a log-normal distribution with mean value and standard deviation equal to $\mu_{pga} = 0.687 \ m/s^2$ and $\sigma_{pga} = 1.312 \ m/s^2$, respectively. The mass of each of the floors is modeled as log-normal random variable with mean value $\mu_M = 180 \times 10^3 \ kg$ and a coefficient of variation of 10%. The Young’s modulus of the steel members and the yield strength are also modeled using a log-normal distribution with coefficient of variation of 10%; the mean values are equal to $\mu_E = 2 \times 10^{11} \ Pa$ and $\mu_{Fy} = 235 \times 10^6 \ Pa$. The constitutive law of the steel is bilinear with strain hardening ratio of 1%. A life time of the structure of 50 years is considered for the analysis. The
discount rate $\lambda$ was set equal to 5%.

The costs of construction of the frame are modeled as a linear function w.r.t. the mass of the structural members [68]. The costs associated with each limit state are taken from the literature [68]. Finally, for the sake of simplicity, the costs of maintenance are not considered.

The structural response of the steel frame depicted in fig. (6) is calculated using the non linear static analysis procedure proposed in FEMA-356 [18], i.e. the so-called coefficient method of displacement modification. In this approach, the structural response due to a seismic event is calculated by relating an idealized pushover curve of the roof displacement of the building and base shear with the elastic response of a SDOF oscillator.

5.2.4. Results

The total number of damage states considered is 7 (see Appendix B). For calculating the probabilities of these damage states, a total of 6 independent reliability analyses are required. These analyses are carried out by means of IS; for each analysis, 400 samples are generated. For each of these 6 reliability analyses (and for each new candidate optimal design $y^r$), the design point associated with the optimization problem in eq. (7) must be identified. As the repeated solution of this optimization problem for the different reliability analyses and different values of the design variables can be numerically expensive, the design point is determined exclusively with respect to the random variable modeling the peak ground acceleration, as this is the most influential variable according to preliminary evaluations performed prior to launching the optimization algorithm. Once the corresponding design points are identified, the aforementioned 6 reliability analyses are used to determine the relation $P[D_i(y, \theta) \geq b]$ (cf. eq. (5)) by means of a monotone cubic interpolation. The parameter $w$ that controls the size of the local optimization domains was set equal to 5. The linear expansion of the normalized demand function (cf. eq. (9)) is performed w.r.t. to the second moment of inertia (principal axis) of the design variables using $N_S = 100$ samples, as in example 1.

The results of the application of the RBO procedure are shown in fig. (7) and Table 2, where $E[C]$ represents expected total costs, $C_C$ denotes construction costs, $E[C_{PD}]$ refers to the expected costs due to partial damage and $MU$ denotes monetary units.

From fig. (7), it can be observed that in the initial design, the costs associated with the damage states largely exceed the costs of construction. As the iterations progress, new candidate optimal designs with construction costs which are higher than the initial one are found. However, for these candidates, the reduction in the costs associated with damage is remarkable. Thus, in the final design, the costs associated with damage are less than the costs of construction, but the overall expected costs are almost half of the costs associated with the initial design. Such results are consistent with other studies in the literature, e.g. [19, 68].
The optimal solution was found after 8 iterations (i.e. 8 local domains are required). The numerical costs associated with each of these iterations is as follows. In each iteration, 6 reliability analyses are performed, i.e. a total of $6 \times 400 = 2400$ samples are required. In addition, for creating a meta-model for reliability, a total of six linear expansions of the normalized demand functions are required (one expansion per each reliability analysis). As there are 4 design variables and $N_S = 100$, then a total of $6 \times 4 \times 100 = 2400$ structural analyses are required for constructing the linear expansions. Therefore, each iteration of the proposed RBO scheme requires a total of 4800 structural analyses.

Finally, it is important to remark that the computational efforts required for solving this problem are considerable, as several non linear static analyses must be performed for estimating probabilities. Therefore, parallel computing techniques were applied in order to solve the RBO problem in acceptable time. This allowed solving the RBO problem in 15 hours, using a total of 20 cores. This reduction in the computational time is remarkable, e.g. assuming linear speed up, approximately 12 days would have been required to solve the same problem sequentially (i.e. using a single core).

6. Conclusions and Outlook

This contribution has introduced a novel approach for solving RBO problems involving discrete design variables. Key issues of the proposed approach are the application of sequential approximations and a meta-model for reliability. Sequential approximations allow dividing the original RBO problem in a series of subproblems, which are easier to solve than the original one. The meta-model for reliability allows solving each of these subproblems with negligible computational efforts, as the meta-model is an explicit function of the design variables.

A salient feature of the approach introduced in this contribution is the meta-model for reliability. This meta-model is constructed in two steps: first, a standard reliability analysis (at the current
candidate optimal design) is carried out; then, the calibration of a linear expansion of the normalized demand function is performed. The second step of the procedure requires some additional structural analyses and does not involve additional reliability analyses. This is a very important advantage over alternative approaches for constructing a meta-model for reliability, e.g., in case the meta-model would be created by fitting a response surface or a neural network to the values of the probability directly, several reliability analyses would be required.

Numerical examples addressed in this contribution indicate that the proposed approach can be very advantageous from a numerical point of view. Although the convergence of the algorithm towards a global or local optimum is not ensured, few iterations are required in order to find improved design solutions. This is a most valuable property, as in problems of interest, engineering criteria allow assessing the quality of a design solution.

Concerning the examples presented and in particular, the second example, it was shown that the proposed approach for RBO can be applied for determining an optimized design considering lifetime losses. This example also showed that for the optimal design of a structure, it is not sufficient to consider just the construction costs, as the costs associated with partial damage play a significant role. Moreover, the example also illustrates that for the solution of involved problems in RBO, parallel computing techniques are necessary in order to reduce computation time to an acceptable level.

Further research efforts will focus on extending the capabilities of the approach reported herein. Issues to be studied are (a) determination of the range of application of the proposed approach, e.g., with respect to the number of design variables, (b) selection of local domains for optimization and (c) construction of the linear expansion of the normalized domain in cases where more than one relevant property of the design variables must be considered.

Acknowledgment

This research was partially supported by the Austrian Research Council (FWF) under Projects No. P17459-N13 and P20251-N13 which are gratefully acknowledged by the authors.

Appendix

A. Stochastic Ground Acceleration Model

The ground acceleration \( (g_A(t)) \) is modeled as a filtered white noise of 15 s of duration. The ground acceleration is calculated as \( g_A(t) = \alpha^T p(t) \); the vectors \( \alpha^T \) and \( p(t) \) are defined as:

\[
\alpha^T = \langle \Omega_1^2, 2\xi_1\Omega_1, -\Omega_2^2, -2\xi_2\Omega_2 \rangle
\]  

(21)
\[
\dot{p}(t) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\Omega_1^2 & -2\xi_1\Omega_1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\Omega_1^2 & 2\xi_1\Omega_1 & -\Omega_2^2 & -2\xi_2\Omega_1
\end{pmatrix} p(t) + \begin{pmatrix}
0 \\
\omega(t)e(t) \\
0 \\
0
\end{pmatrix}
\] (22)

where \(\Omega_1 = 15.6 \text{ rad/s}, \xi_1 = 0.6, \Omega_2 = 1.0 \text{ rad/s}\) and \(\xi_2 = 0.9\) are the filter parameters; \(\omega(t)\) denotes a white noise signal and \(e(t)\) is an envelope function.

\[
e(t) = \begin{cases}
t^2/16 & \text{if } t \in [0, 4 \text{ s}] \\
1 & \text{if } t \in [4 \text{ s}, 10 \text{ s}] \\
e^{(t-10)^2} & \text{if } t \in [10 \text{ s}, 15 \text{ s}]
\end{cases}
\] (23)

A time discretization step equal to \(\Delta t = 0.01 \text{ s}\) is used to model the ground acceleration. Thus, the discrete representation of the white noise signal is \(\omega(t_z) = \sqrt{2\pi S/\Delta t} \theta_z, z = 1, \ldots, 1501\), where \(S = 0.8 \times 10^{-4} \text{ m}^2/\text{s}^3\) is the spectral density of the white noise and \(\theta_z, z = 1, \ldots, 1501\) are independent, identically distributed standard Gaussian random variables.

### B. Quantification of Damage and its Probability of Occurrence

In the approach proposed in [68], the storey drift ratio \(\Delta\) is related with the level of damage as indicated in Table 3; note that Table 3 does not refer to collapse only, but also to intermediate damage states. It should be noted as well that the storey drift ratio is a random quantity, which is a function of both the design variables and the uncertain parameters, i.e. \(\Delta = \Delta(y, \theta)\).

The probability of occurrence of the \(i\)-th limit state is calculated by means of the following formula:

\[
p_{F,i} = P[\Delta(y, \theta) \geq \Delta_i] - P[\Delta(y, \theta) \geq \Delta_{i+1}] \tag{24}
\]

where \(\Delta(\cdot, \cdot)_i\) and \(\Delta(\cdot, \cdot)_{i+1}\) are the drift ratios defining the lower and upper bounds of the \(i\)-th damage state and where \(P[\Delta(y, \theta) \geq \Delta_i]\) is the exceedance probability given occurrence. As the occurrence of seismic events is modeled using a Poisson process, \(P[\Delta(y, \theta) \geq \Delta_i]\) can be calculated as [68]:

\[
P[\Delta(y, \theta) \geq \Delta_i] = -\frac{1}{\nu t} \ln \left(1 - P_t[\Delta(y, \theta) \geq \Delta_i]\right) \tag{25}
\]

where \(P_t[\Delta(y, \theta) \geq \Delta_i]\) is the exceedance probability within a period \((0, t)\). This last probability is defined as a multi-dimensional integral, i.e.:

\[
P_t[\Delta(y, \theta) \geq \Delta_i] = \int_{\frac{\Delta(y, \theta)}{\Delta_i} \geq 1} f(\theta) d\theta \tag{26}
\]
This integral can be evaluated using any appropriate reliability technique.

References


[44] M. Papadrakakis and N.D. Lagaros. Reliability-based structural optimization using neural net-


List of Tables

Table 1
Results of example 1

Table 2
Results of example 2

Table 3
Definition of failure criteria and damage states

<table>
<thead>
<tr>
<th>Initial design</th>
<th>Final design</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>W12x136</td>
</tr>
<tr>
<td>$y_2$</td>
<td>W12x136</td>
</tr>
<tr>
<td>$y_3$</td>
<td>W12x120</td>
</tr>
<tr>
<td>$y_4$</td>
<td>W12x120</td>
</tr>
</tbody>
</table>

| $p_F$          | $5.6 \times 10^{-3}$ |
| Volume [m$^3$]| $4.42$ |
|               | $2.49$ |

Table 1: Results of example 1

<table>
<thead>
<tr>
<th>Initial design</th>
<th>Final design</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>W14x145</td>
</tr>
<tr>
<td>$y_2$</td>
<td>W14x82</td>
</tr>
<tr>
<td>$y_3$</td>
<td>W14x82</td>
</tr>
<tr>
<td>$y_4$</td>
<td>W14x82</td>
</tr>
</tbody>
</table>

| $E[C] [MU]$   | $8.3 \times 10^6$ |
| $C_C [MU]$   | $2.1 \times 10^6$ |
| $E[C_{PD}] [MU]$ | $4.1 \times 10^6$ |

Table 2: Results of example 2

<table>
<thead>
<tr>
<th>Failure criterion</th>
<th>Damage state</th>
<th>Drift ratio [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>none</td>
<td>$\Delta \leq 0.2$</td>
</tr>
<tr>
<td>2</td>
<td>slight</td>
<td>$0.2 &lt; \Delta \leq 0.5$</td>
</tr>
<tr>
<td>3</td>
<td>light</td>
<td>$0.5 &lt; \Delta \leq 0.7$</td>
</tr>
<tr>
<td>4</td>
<td>moderate</td>
<td>$0.7 &lt; \Delta \leq 1.5$</td>
</tr>
<tr>
<td>5</td>
<td>heavy</td>
<td>$1.5 &lt; \Delta \leq 2.5$</td>
</tr>
<tr>
<td>6</td>
<td>major</td>
<td>$2.5 &lt; \Delta \leq 5.0$</td>
</tr>
<tr>
<td>7</td>
<td>destroyed</td>
<td>$\Delta &gt; 5.0$</td>
</tr>
</tbody>
</table>

Table 3: Definition of failure criteria and damage states